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Mass and Charge from Higher Dimensional Geometry

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We explore ways in which phenomenological physical quantities such as the rest mass and electric charge of a particle could be explained as properties of higher dimensional geometry. In 5D, it is shown that mass is related to the extra coordinate and charge is related to the extra momentum. This approach can be extended to supergravity and string theory.

1. INTRODUCTION

Quantities such as the rest mass m and electric charge q of a test particle are still regarded as phenomenological, in the sense that we have no account of their origin from fundamental theory. However, noble attempts to explain quantities such as m and q have been made through the history of physics. Mach's principle represented such an attempt, and still generates interesting discussion (Barbour and Pfister, 1995). Einstein's 4D general relativity is widely regarded as not properly incorporating Mach's principle, but in its spirit Einstein in his later work continued to espouse the view that physics ought to be derived from geometry, as, for example, in 5D Kaluza–Klein theory [for a recent review of this see Overduin and Wesson (1997)]. A related, though alternative view is that we should aim to derive $m = m(x^{\alpha})$, where $\alpha = 0$–3, from a scale-covariant extension of general relativity (Hoyle and Narlikar, 1974). This idea has merit because it is computational rather than conceptual. However, while to most workers this is an advantage, we should not forget that symbols like m and q were introduced into physics a

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long time ago as convenient but somewhat ill-defined concepts (Jammer, 1961). To that extent, they involve a certain element of subjectivity, as pointed out by Eddington (1939; see also Kilmister, 1994). The same comment applies to other quantities in common use in modern physics, such as the so-called fundamental constants (Wesson, 1992). Ambiguity in our concepts can be handled to a certain extent, but leads to puzzling physics. For example, in gravity we have to make a logical distinction between the inertial and active and passive gravitational masses of a particle, even though experiments to test the equivalence principle show that they are equal to remarkable accuracy (Will, 1993). And when we sum over a number of particles and include their interactions, we are led to several acceptable but inequivalent definitions of the mass of a macroscopic body and well-known problems to do with the (non)localizability of gravitational energy (Hayward, 1994). In particle physics, there is no practical distinction between the mass of a particle and the energy of a resonance. And the electric charge of one particle depends on the energy of interaction or distance of approach of another one (this is commonly attributed to vacuum polarization, a mechanism which is supported by measurements of the variability of the fine-structure "constant," but would have been anathema to Einstein). We conclude that symbols such as m and q are ill-defined, and that if they are to continue to be used in physics, they require some deeper justification than they have been given before.

Recently, there has been remarkable progress in explaining phenomenological properties of matter, such as the density ρ and pressure p of a fluid, as consequences of higher dimensional geometry. The basic idea is to rewrite the 10 4D Einstein equations with matter as a subset of the 15 5D Kaluza–Klein equations in vacuum. In this way, it is possible to derive $\rho = \rho(x^a)$, $p = p(x^a)$, where $a = 0$–4, thereby explaining matter as a consequence of pure geometry. This approach is not restricted to 5D, but as the algebraically simplest extension there has sprung up a considerable literature on this space-time-matter theory (for a review see Wesson *et al.*, 1996). It differs from previous versions of Kaluza–Klein theory in that the metric can depend on all five coordinates, the extra dimension is not necessarily compactified, and the reduction to 4D occurs via a hypersurface condition related to a solution of the 5D geodesic equation (Mashhoon *et al.*, 1994). This theory agrees with the classical tests of relativity, and while it is not our intention to justify it here, its success does suggest that there may be a way to connect the rest mass of a particle to the geometry of a higher dimensional space. We explore this possibility in what follows.

2. MASS, CHARGE, AND GEOMETRY

In this section we will draw on results proved in the references quoted in the preceding section, to see how far the rest mass of a particle m, and also its electric charge q, may be related to geometry. We will proceed in accordance with Occam's razor, concentrating on the 5D extension and adding assumptions as to interpretation only as they are required.

Let us consider dynamics in 4D and 5D, assuming that the former has to be embedded in the latter, but using only the metric and neglecting constraints that might derive from a set of field equations.

In 4D, we have a line element that is given in terms of a metric tensor by $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$. (Throughout, Greek letters will run 0–3 and Latin letters 0–4; and we will usually absorb physical constants via units wherein the speed of light, Newton's constant, and Planck's constant are $c = 1$, $G = 1$, $h = 1$, respectively.) The Lagrangian is commonly defined as

$$\mathcal{L} = m\frac{ds}{d\lambda} + qA_\mu \frac{dx^\mu}{d\lambda} \tag{1}$$

where λ is a parameter and A_μ is the electromagnetic potential. This is acceptable, but clearly m and q are introduced ad hoc. Overlooking this for the moment, we form the momentum 4-vector:

$$p_\alpha = \frac{\partial\mathcal{L}}{\partial(dx^\alpha/d\lambda)} = mg_{\alpha\beta}\frac{dx^\beta}{ds} + qA_\alpha \tag{2}$$

That is, $p_\alpha = mg_{\alpha\beta}u^\beta + qA_\alpha$ and $p^\alpha \equiv g^{\alpha\beta}p_\beta = mu^\alpha + qA^\alpha$, where $u^\alpha \equiv dx^\alpha/ds$ is the 4-velocity. The product is

$$p_\alpha p^\alpha = m^2 u_\alpha u^\alpha + 2mqA_\alpha u^\alpha + q^2 A_\alpha A^\alpha \tag{3}$$

which is just $+m^2$ when we choose the signature to be $g_{\alpha\beta} = (+1, -1, -1, -1)$ and put $q = 0$. Locally $g_{\alpha\beta} \simeq \eta_{\alpha\beta}$, and for a charged particle moving with a 3-velocity $v^{123} = dx^{123}/dt$ we have

$$p^0 = m\frac{dt}{ds} + qA^0 = \frac{m}{(1 - v^2)^{1/2}} + qA^0 \tag{4a}$$

$$p^{123} = m\frac{dx^{123}}{ds} + qA^{123} = \frac{mv^{123}}{(1 - v^2)^{1/2}} + qA^{123} \tag{4b}$$

which we call the energy and (3D) momenta.

The preceding is of course a covariant formulation of a special-relativity, no-charge argument that is simplistic, but instructive. It consists in writing

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) \tag{5a}$$

$$1 = \left(\frac{dt}{ds}\right)^2 - \left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] \tag{5b}$$

$$1 = \left(\frac{E}{m}\right)^2 - \left(\frac{p}{m}\right)^2 \tag{5c}$$

That is, Minkowski space plus the definitions $u^0 \equiv E/m$ and $u^{123} \equiv p^{123}/m$ result in the relation $E^2 = p^2 + m^2$. This is the basis of particle physics, and is in excellent agreement with experimental data. But, as pointed out above, the introduction of m is purely ad hoc. We should aspire to explain the origin of m, and relations (1)–(5), in terms of some more sophisticated theory. We will attempt this below, but as an intermediate step, let us see what algebraic opportunities occur if we extend the geometry.

In 5D, we have a line element that is given in terms of a metric tensor by $dS^2 = g_{ab}\, dx^a\, dx^b$. As in other versions of 5D (Kaluza–Klein) theory, we choose to write this as a sum of a 4D part ($ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$) and an extra part that depends on a scalar field (ϕ) and a vector field (A_α):

$$dS^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta - \Phi^2(d\ell + A_\alpha\, dx^\alpha)^2 \tag{6}$$

This split is of course motivated by the Kaluza–Klein "miracle," wherein the field equations in vacuum for metric (6) split naturally into Einstein's equation for $g_{\alpha\beta}$, Maxwell's equation (for A_α), and a scalar wave equation (for Φ). We note in passing that while we will employ (6) because we wish to gain insight eventually into the nature of electric charge, it is historically conditioned by our development of gravity and electromagnetism as two different subjects. The strength of Kaluza–Klein theory lies actually in the fact that these interactions may be mixed by coordinate transformations: it is a truly unified theory, and while (6) is convenient, it is not unique. Having made this observation, we note that with the metric in the form (6), the Lagrangian is commonly defined as

$$\mathcal{L} \equiv m\,\frac{dS}{d\lambda} = m\left[g_{\alpha\beta}\,\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda} - \Phi^2\left(\frac{d\ell}{d\lambda} + A_\alpha\,\frac{dx^\alpha}{d\lambda}\right)^2\right]^{1/2} \tag{7}$$

Here m is still introduced ad hoc, but the theory has one extra parameter associated with the fifth coordinate ℓ, and the charge q does not appear explicitly (see below). From (7) we can form the 4-part of the momentum 5-vector:

$$\hat{p}_\alpha = \frac{\partial\mathcal{L}}{\partial(dx^\alpha/d\lambda)} = m\left[g_{\alpha\beta}\,\frac{dx^\beta}{dS} + BA_\alpha\right] \tag{8a}$$

$$B \equiv -\Phi^2\left(\frac{d\ell}{dS} + A_\alpha\,\frac{dx^\alpha}{dS}\right) \tag{8b}$$

This is the 5D analog of the 4D relation (2). The extra or fifth component is

$$\hat{p}_4 = mB \tag{8c}$$

We see that the scalar function B of (8b) is just the momentum in the extra dimension (this was known in a simple form to Kaluza, Klein, and others, but the above derivation is general). We might expect B to be small in many situations, because from (6) and (8b) we have

$$\frac{ds}{dS} = \left[1 + \frac{B^2}{\Phi^2}\right]^{1/2} \tag{9}$$

so $ds \simeq dS$ in this case. To get the contravariant forms of the covariant vectors (8a) and (8c), we note that

$$g^{ab} = \begin{bmatrix} g^{\alpha\beta} & -A^{\alpha} \\ -A^{\alpha} & (A^{\mu}A_{\mu} - \Phi^{-2}) \end{bmatrix} \tag{10}$$

with $A^{\mu} \equiv g^{\mu\nu}A_{\nu}$. This gives $\hat{p}^{\alpha} \equiv g^{\alpha b}\hat{p}_b = m\hat{u}^{\alpha}$ and $\hat{p}^4 \equiv \hat{g}^{4b}\hat{p}_b = m\hat{u}^4$, where $\hat{u}^a \equiv dx^a/dS$ is the 5-velocity. The product is

$$\hat{p}_{\alpha}\hat{p}^{\alpha} = m^2 \tag{11}$$

This is the 5D analog of the 4D relation (3). However, q does not appear in the former, whereas it does in the latter, so we need to inquire where this parameter fits into the extended geometry. To answer this, we rewrite (8a):

$$\hat{p}_{\alpha} = m[g_{\alpha\beta}\hat{u}^{\beta} + BA_{\alpha}] \tag{12}$$

Although it is the 4-part of a 5-vector, the metric (6) from which it is derived is invariant under the 4D subset of general 5D coordinate transformations $x^{\alpha} \to \bar{x}^{\alpha}(x^{\beta})$, and so therefore (12) must be also. That is, $\hat{p}_{\alpha}$ is a 4-vector. To recast it in 4D form, we use (9) and find

$$\hat{p}_{\alpha} = \left[m\left(g_{\alpha\beta}u^{\beta} + \frac{B}{(1 + B^2\Phi^{-2})^{1/2}}A_{\alpha}\right)\right]\frac{ds}{dS} \tag{13}$$

This can be compared with the purely 4D relation (2):

$$p_{\alpha} = m\left(g_{\alpha\beta}u^{\beta} + \frac{q}{m}A_{\alpha}\right) \tag{14}$$

Clearly the vectors are related via $\hat{p}_{\alpha} = p_{\alpha}(ds/dS)$ and the charge/mass ratio of the test particle is

$$\frac{q}{m} = \frac{B}{(1 + B^2\Phi^{-2})^{1/2}} \tag{15}$$

Since in (8b) and (6) B and Φ will in general depend on x^a, the charge will in general be a function of coordinates times the mass. To fix q exactly would need Φ and A_{μ} to be determined by field equations and $\hat{u}^a$ to be determined by the Lagrange equations, or equivalently the geodesic equations.

The latter with 5D Christoffel symbols are

$$\frac{d^2x^a}{dS^2} + \Gamma^a_{bc}\frac{dx^b}{dS}\frac{dx^c}{dS} = 0 \tag{16}$$

and have been much studied in the literature, along with associated solutions of the field equations. [In terms of the Ricci tensor these are usually taken to be given by $R_{ab} = 0$; see Overduin and Wesson (1997) for a review.] We do not wish to repeat what is known about (16) here, but we do wish to make a few comments relevant to what we have done above.

The extra component of (16) can be written in once-integrated form using the scalar B of (8b) as

$$\frac{dB}{dS} = \frac{1}{2}\frac{\partial g_{ab}}{\partial \ell}\frac{dx^a}{dS}\frac{dx^b}{dS} \tag{17a}$$

$$B \equiv -\Phi^2(\hat{u}^4 + A_\alpha \hat{u}^\alpha) \tag{17b}$$

Thus B is a constant of the motion if the 5-metric is independent of $x^4 = \ell$. Then by (8.3), $\hat{p}_4 = mB$ is a constant momentum in the extra dimension. However, even in this case if the scalar potential $g_{44} = -\Phi^2$ depends on x^α, the q/m ratio of a particle (15) will not be a constant. This means that the charge of a test particle can vary in ordinary (3D) space, providing in principle an alternative to the mechanism of vacuum polarization mentioned in Section 1.

The 3D components of (16) in general consist of terms identical to those of geodesic motion in the Einstein sense, plus scalar-field terms and electromagnetic terms. The latter include the Lorentz force, provided q/m for a test particle is identified as in (15). That is, the 5D equations of motion include those we are familiar with from general relativity and classical electromagnetism.

The zeroth or time component of (16) can be written in once-integrated form using a scalar function C as

$$\frac{dC}{dS} = \frac{1}{2}\frac{\partial g_{ab}}{\partial t}\frac{dx^a}{dS}\frac{dx^b}{dS} \tag{18a}$$

$$C \equiv \frac{g_{00}^{1/2}}{(1 - v^2)^{1/2}}\left(1 + \frac{B^2}{\Phi^2}\right)^{1/2} + BA_0 \tag{18b}$$

Thus C is a constant of the motion if the 5-metric is independent of $x^0 = t$. This expression is similar to the 4D one for the energy E of a test particle in a static spacetime

$$E = \frac{g_{00}^{1/2}}{(1 - v^2)^{1/2}}m + qA_0 \tag{19}$$

We will make a detailed comparison of these relations below, but we note here that they are algebraically compatible.

The 5D geodesic equation (16) splits naturally into a 4D part and an extra part when the metric has the form called canonical (Mashhoon *et al.*, 1994):

$$dS^2 = \left(\frac{\ell}{L}\right)^2 g_{\alpha\beta}(x^\gamma, \ell)\, dx^\alpha\, dx^\beta - d\ell^2 \tag{20}$$

Here L is a constant introduced for dimensional consistency, which would be identified with some physical parameter in an exact solution of the field equations $R_{ab} = 0$. An example is the 5D Schwarzschild–de Sitter solution

$$dS^2 = \left(\frac{\Lambda\ell^2}{3}\right)\left[\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)dt^2 - \frac{dr^2}{1 - 2M/r - \Lambda r^2/3} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] - d\ell^2 \tag{21}$$

where $L = (3/\Lambda)^{1/2}$ is related to the cosmological constant. We do not wish to indulge in a detailed discussion of metrics with form (20), but do wish to note a couple of things. First, any 5D metric can be put into form (20) in principle, because the five coordinate degrees of freedom can be used to set $g_{0\alpha} = 0$, $g_{44} = -1$. However, this may not be convenient in practice, since the previously overt electromagnetic and scalar interactions become "hidden" in the spacetime part of the metric. Second, the real significance of metrics with form (20) is when $\partial g_{\alpha\beta}/\partial\ell = 0$. To appreciate this, we can write the $a = 0123$ and $a = 4$ components of the geodesic (16) for metric (20):

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \left(-g^{\mu\alpha} + \frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}\right)\frac{d\ell}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial\ell} \tag{22a}$$

$$\frac{d^2\ell}{dS^2} - \frac{2}{\ell}\left(\frac{d\ell}{dS}\right)^2 + \frac{\ell}{L^2} = -\frac{1}{2}\left[\frac{\ell^2}{L^2} - \left(\frac{d\ell}{ds}\right)^2\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial\ell} \tag{22b}$$

We see that when $\partial g_{\alpha\beta}/\partial\ell = 0$, the motion in 4D is geodesic in the Einstein sense. The motion in the extra dimension is solved by $\ell = \ell_0/\cosh[(s - s_0)/L] \sim \ell_0 e^{-s/L}$ ($s >> s_0$, L, where s_0 and ℓ_0 are constants). That is, the motion in 4D is identical to what it is in general relativity, and the motion in the extra dimension is damped toward the hypersurface $\ell = 0$ asymptotically.

So far, we have derived q as a function of coordinates times m, but we have yet to tackle the more fundamental question of the nature of the latter parameter. It may be wise before proceeding to review the argument to this stage and where we hope it will lead. The 4D Lagrangian (1) introduces m and q ad hoc, as constant coefficients for separate interactions due to gravity and electromagnetism. This leads to relations (2)–(5), which are experimentally verified, at least in the limit of local physics. The 5D line interval (6) introduces m ad hoc but not q. This is clearly a significant step forward, and the ensuing relations (8)–(14) lead to an expression for the charge/mass ratio of a particle (15) that is essentially geometric in nature: q/m is defined at every point of a 5D manifold, and can in principle be evaluated given solutions of the field equations and the geodesic equations. The latter are (16), and have a fifth and time components (17) and (18) that are associated with charge and energy. Both these components, and the 3-space components, agree with known physics, including in particular the 4D energy (19). However, any 5D metric can be put into the so-called canonical form (20), and many solutions of the field equations have this form, such as the basic Schwarzschild–deSitter solution (21). The geodesic equations (16) for the canonical metric result in relations (22) for the spacetime dimensions and the extra dimension that have interesting properties, including a natural selection of 4D geodesic motion and evolution to a hypersurface.

There are several aspects of the preceding argument that are worthwhile noting before proceeding. For example, the 5D interval (6) is devoid of constants, while the m which is attached to the associated Lagrangian (7) is a constant that multiplies the whole expression: it gives an action with the conventional 4D physical dimensions, but serves no algebraic purpose. The fact that the analysis based on the Lagrangian (7) gives the same result as that based on the metric (6) and geodesic (16) for the ratio q/m is due to the innocuous way that m is introduced. A related observation is that in 4D general relativity, where the geodesic gives accelerations and not forces, there is no way to introduce m except as a constant. It cannot be derived. In 5D Kaluza–Klein theory, the situation is more promising. For, besides being a unified theory of gravity and electromagnetism that yields the ratio q/m, the extension of the geometry necessarily includes a scalar interaction which we can in principle employ to explain the origin of rest mass (we can view it as the classical analog of the Higgs field of quantum field theory). The extension of the geometry also raises the option of identifying historical 4D physical quantities as manifestations of 5D physics on (or close to) a 4D hypersurface. Of course, any 5D manifold with an extra coordinate ℓ will define a 4D manifold on a hypersurface $\ell = \ell_0$ (say). In general, the net result of interpreting 5D physics on a hypersurface is to introduce a constant ℓ_0 into 4D physics. In addition, in the local weak-field (or special relativity) limit of such a theory, the velocity in the extra dimension will enter as a parameter into 4D physics. It is reasonable to expect that if the local limit of a 5D theory introduces in this way two geometrical parameters, then they be related to 4D physical quantities such as m and q. The question is not whether we can explain m and q in principle, but whether, given the constraints of known physics, we can do it in practice.

A basic question to be answered before we get to technicalities is whether mass can in fact be geometrized. The answer to this is in the affirmative (Hoyle and Narlikar, 1974; Wesson, 1992). Nature provides us with parameters such as the speed of light c, Newton's constant of gravity G, and Planck's constant of action h. These allow us to write

$$x_m \equiv \frac{Gm}{c^2} \qquad \text{or} \qquad x_m \equiv \frac{h}{mc} \tag{23}$$

depending on whether we use gravitational or atomic units. The question of whether G or h (or c) is constant is irrelevant, as only the noted combinations of symbols have geometrical significance. This is also why we can choose (Planck) units with $c = 1$, $G = 1$, and $h = 1$. The important thing is that the rest mass can be regarded as a length if we so desire. (To this extent, the above is just the analog of regarding ct or time as a length.) Charge can be treated this way also, but even at this elementary level a difference becomes apparent: we can only geometrize q by including the gravitational constant G, via $x_q \equiv \sqrt{G}c^{-2}q$. Equivalently, the combination $e^2/\hbar c$ which defines the fine-structure constant for an (asymptotic, large-range) value of the electron charge e is dimensionless. This is presumably connected to the trite but irrefutable fact that charged particles have mass, but not the other way around. The implication is that mass is more fundamental than charge. This conclusion agrees with what we derived previously on the basis of 5D geometry: q/m as defined by (15) allows us to set q to zero while leaving m finite.

Technically, the expression of the 4D phenomenological parameters m and q in terms of 5D geometrical quantities like x^a and $\hat{u}^a$ leaves little room for maneuvre. This can most readily be seen by a detailed comparison of the 5D quantity (18b) with the 4D quantity (19).

Let us first look at these expressions for the case of an uncharged test particle (q, $B \to 0$). In (18b), g_{00} refers to a 5D metric coefficient, and for illustration we take the basic metric (21). Reintroducing L and considering the weak-field limit $L >> r >> 2M$, we obtain the metric $dS^2 \simeq (\ell/L)^2\, ds^2 - d\ell^2$, where $ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$ and $\eta_{\alpha\beta} = (+1, -1, -1, -1)$, so $g_{00}^{1/2} = \ell/L$. In (19), g_{00} refers to a 4D metric coefficient, so in the corresponding limit $g_{00}^{1/2} = 1$. Then we obtain

$$C = \frac{\ell}{L(1 - v^2)^{1/2}} \tag{24a}$$

$$E = \frac{m}{(1 - v^2)^{1/2}} \tag{24b}$$

Here C is dimensionless, while E has the dimensions of a length (due to the ad hoc introduction of m). Thus we make the identification

$$E \equiv CL = \frac{\ell}{(1 - v^2)^{1/2}} \tag{25}$$

We see that the role of the 4D rest mass m is played in 5D geometry by the coordinate $x^4 = \ell$.

This may seem surprising, but when we examine possible objections they resolve themselves. For example, on a hypersurface $\ell = \ell_0$ of a 5D metric we can in principle always identify the 4D parameter m with ℓ_0, but there is no guarantee that the motion in the fifth dimension will remain close to this hypersurface, so in general 4D rest mass ought to be variable. However, for many metrics the motion does evolve to a hypersurface asymptotically; and for metrics in the canonical form whose spacetime components do not depend on x^4, we saw in (22) that the 4D motion is perfectly geodesic, so even if rest masses do vary (slowly), *there is no way to detect this dynamically.* Another possible objection is that nature gives us two dimensionally consistent choices for x^4, namely $x_4 = m$ and $x^4 = 1/m$ of (23). The present analysis implies the first, so we might worry about the status of the second. However, the 5D theory is fully covariant, so $x^4 = \ell$ and $x^4 = 1/\ell$ (or any other choice) are admissible. This latter observation does, though, raise an important point: by 5D coordinate transformations we can change the form of any metric, and it is *only* in the case where the metric is in the canonical form (or close to it) that we can make the identification (25) with $x^4 = \ell = m$. (Another way to see this is to note that the 4D action of particle physics $m \int ds$ only agrees formally with the 5D action of Kaluza–Klein theory $\int dS = \ell_0 \int ds$ when we are on the hypersurface and the metric is in canonical form.) This is analogous to what happens in 4D, where we commonly identify quantities like the energy and momenta of a particle only in coordinate frames that are close to the special-relativity form. Our conclusion is that there is a special (canonical) system of coordinates where we can write $x^4 = m$, but in general we should regard x^4 as a mass-related coordinate rather than the mass itself.

Let us now look at (18b) and (19) for a charged test particle in the weak-field limit. A comparison of these relations shows that we now have to make a distinction between the mass of an uncharged particle (which we have above labeled m) and the mass of a charged particle (which we here label m_q). The appropriate definitions are

$$m_q \equiv \ell\left(1 + \frac{B^2}{\Phi^2}\right)^{1/2} \tag{26a}$$

$$q \equiv B\ell \tag{26b}$$

The ratio of these gives a charge/mass ratio that agrees exactly with the ratio (15) found from a Lagrangian approach and the coefficient of the Lorentz force term in the spatial components of the geodesic (16). This is of course as expected, since the mass of a charged particle or the mass that feels the Lorentz force should include both the neutral rest-mass energy and the energy of the charge. Again, however, the identifications (26) presume a special (canonical) system of 5D coordinates.

If we use instead of the canonical form of the metric (20), a 5D analog of Minkowski space (5), then

$$dS^2 = dt^2 - (dx^2 + dy^2 + dz^2) - d\ell^2 \tag{27}$$

and $A_\alpha = 0$, $\Phi^2 = 1$ in (6), so $B = -d\ell/dS$ by (8b) and $ds/dS = (1 + B^2)^{1/2}$ by (9). These relations with (27) allow us to define a mass

$$m_{vw} = \frac{m_{00}}{(1 - v^2 - w^2)^{1/2}} = m_{00}\frac{dt}{dS} \tag{28}$$

which boosts a constant m_{00} by velocities v, w in ordinary space and the extra dimension. For $v \neq 0$ and $w = 0$ we have $m_{v0} = m_{00}(1 - v^2)^{-1/2}$, agreeing with the usual definition of the energy. For $v = 0$ and

$$w \equiv d\ell/dt = d\ell/ds = (d\ell/dS)(dS/ds) = -B(1 + B^2)^{-1/2} \neq 0$$

we have $m_{0w} = m_{00}(1 - w^2)^{-1/2} = m_{00}(1 + B^2)^{1/2}$, agreeing with the above definition (26a) for the charged mass. This argument, while it does not identify the "raw" rest mass because the metric is not in an appropriate form, confirms that charge is due to motion in the extra dimension and that the mass is augmented thereby as expected from an extra Lorentz boost.

To this point, our considerations have involved classical dynamics, because it is from the paths of test particles in spacetime that their masses and charges are determined in practice. To conclude, however, we wish to make some observations involving quantum mechanics and the field equations that bear on our problem and may be of interest in theory.

In 5D Kaluza–Klein theory, as mentioned before, the 15 field equations split naturally into sets of 10, 4, and 1. The last is a wave equation in the scalar potential $g_{44} = -\Phi^2$. If the field equations in terms of the 5D Ricci tensor are $R_{ab} = 0$, the component $R_{44} = 0$ for a convenient form of the metric yields

$$\Box\Phi \equiv g^{uv}\Phi_{,\mu;\nu} = \frac{1}{\Phi}\left[\frac{g^{\lambda\beta}_{,4}g_{\lambda\beta,4}}{4} + \frac{g^{\lambda\beta}g_{\lambda\beta,4,4}}{2} - \frac{\Phi_{,4}g^{\lambda\beta}g_{\lambda\beta,4}}{2\Phi}\right] \tag{29a}$$

Here a comma denotes the partial derivative and a semicolon denotes the

(4D) covariant derivative. The above relation bears some resemblance to the Klein–Gordon equation

$$\Box\psi = m^2\psi \tag{29b}$$

which is the relativistic wave equation for a spinless particle of mass m. If there were some argument for relating the scalar potential Φ of (29a) to the wave function ψ of (29b), a comparison would define the rest mass m in terms of the geometry of a 5D manifold.

In 5D Kaluza–Klein theory, the 5D Ricci scalar R is perforce zero if the field equations are $R_{ab} = 0$. The theory is to this extent inherently scale-free in 5D. However, the 4D Ricci scalar ${}^{(4)}R$ is *not* in general zero, and in the notation introduced above has the value

$${}^{(4)}R = -\frac{1}{4\Phi^2}\left[g^{\mu\nu}_{,4}g_{\mu\nu,4} + (g^{\mu\nu}g_{\mu\nu,4})^2\right] \tag{30a}$$

That this is in general finite means that hypersurface conditions introduce scales in 4D. This raises an interesting option, analogous to spontaneous symmetry breaking in quantum field theory. It is simply that a 5D space can contain a curved 4D subspace with a standing matter wave which, by Planck's law, defines a particle of mass m where

$$m = \frac{h\sqrt{{}^{(4)}R}}{c} \tag{30b}$$

in physical units. This would define the rest mass m in terms of the local curvature of the 4D part of a 5D manifold.

The ideas presented in the two preceding paragraphs need to be investigated in detail. Both involve an amalgam of quantum mechanics and field theory, and an objection to both is that there is not yet unanimity about the field equations of the underlying theory. For this reason, we have concentrated in the above on dynamics as derived from the Lagrangian or the geodesic. (Our view is that while field equations may be a subject of discussion, it is difficult to conceive of any way to derive the paths of particles other than from an action principle.) But there is not necessarily any conflict between the dynamics we have discussed in the bulk of this section and the ideas we have presented latterly, and in a complete theory the dynamics and field equations would be consistent.

3. SUMMARY AND DISCUSSION

Physics has traditionally proceeded by formulating equations which, though experimentally verifiable, contain parameters which are introduced ad hoc. The rest mass and electric charge of a test particle are prime examples. Progress has often occurred through explaining such ad hoc parameters in terms of some more sophisticated theory. Mach's principle, though it exists in various versions, is basically a statement of the desire to write $m = m(x^a)$ where x^a denotes a general set of coordinates ($a = 0, 1, 2, 3, \ldots$). The same motivation, though it has gained less attention, applies to $q = q(x^a)$. Some progress with the latter problem was made by Klein, who, following Kaluza, realized that gravity (involving m) and electromagnetism (involving q) can be treated as parts of a unified approach to physics based on 5D geometry. This was, of course, an extension of 4D general relativity, and as such was endorsed by Einstein. However, the algebraic complexity of 5D Riemannian geometry invited restrictions, notably the cylinder condition (which banned dependence of the metric on the extra coordinate) and compactification (which hobbled the topology of the extra dimension). These led to physical shortcomings to do with the masses of elementary particles and the value of the cosmological constant. Only recently has it become clear that if we relax these artificial conditions, we arrive at a 5D theory of gravity, electromagnetism, and a scalar field that contains 4D general relativity and agrees with its classical tests. This unrestricted version of 5D general relativity is already known to generate expressions for the density $\rho = \rho(x^a)$ and pressure $p = p(x^a)$ of a fluid which explain these 4D phenomenological quantities in terms of 5D geometry. What we have done in the present work is to extend this to the properties of discrete particles.

If the world is 5D in nature, we can write its line element (6) as a 4D part (general relativity) plus parts that involve a scalar field and electromagnetism:

$$dS^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta - \Phi^2(d\ell + A_\alpha\, dx^\alpha)^2$$

There is a quantity (8b) associated with this metric which is not present in general relativity and depends among other things on the velocity in the extra dimension:

$$B \equiv -\Phi^2\left(\frac{d\ell}{dS} + A_\alpha \frac{dx^\alpha}{dS}\right)$$

It is a constant of the motion if the 5D metric is independent of $x^4 = \ell$, but has significance otherwise. Another quantity (18b) associated with the metric *is* present in general relativity in a simpler form, and depends on the velocity in ordinary (3D) space:

$$C \equiv \frac{g_{00}^{1/2}}{(1 - v^2)^{1/2}}\left(1 + \frac{B^2}{\Phi^2}\right)^{1/2} + BA_0$$

It is a constant of the motion if the 5D metric is independent of $x^0 = t$. A study of dynamics based on the Lagrangian or the geodesic shows that B is related to charge and C is related to energy (including rest mass). Of these, the latter is more fundamental. In an appropriate (canonical) system of coordinates in the weak-field limit, the energy or mass of a neutral particle (25) is

$$E = \frac{\ell}{(1 - v^2)^{1/2}}$$

The rest mass of a charged particle (26a) is

$$m_q = \ell\left(1 + \frac{B^2}{\Phi^2}\right)^{1/2}$$

and its charge (26b) is

$$q = B\ell$$

The ratio q/m_q which appears in the Lorentz law and figures prominently in particle physics is

$$\frac{q}{m_q} = \frac{B}{(1 + B^2\Phi^{-2})^{1/2}}$$

and is in general coordinate-dependent, as are m_q and q. Modulo covariant versions of the 5D metric, the conclusions to be drawn from the equations quoted in this paragraph are that rest mass is related to the extra coordinate and electric charge is related to the extra momentum.

The above results have been derived on the basis of 5D geometry, but we are under no illusions that this is a unique choice of dimensionality. It just so happens that m and q can be accommodated by this (minimal) extension of general relativity. This is presumably because both parameters are mechanical in nature. If we wish to include other parameters such as the ones used in particle physics, we expect to need higher order manifolds. In N-dimensional Riemannian field theory N is promiscuous, to be chosen in the most convenient way necessary to explain a given physical situation. In a way, what we have done in the present work is to outline a mode of interpretation that can be applied *in extenso* to supergravity and string theory. As long as we follow the legacy of Einstein, the overriding principle is that all physical parameters have to be expressible in geometrical terms.

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